Faculty of Mechanical Engineering
Universiti Teknologi Malaysia

Project Report

MMJ 1113 Computational Methods for Engineers

Project 1

by

Awang Selamat
XYZ007

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1 Statement of the Problem

A bungee jumper with a mass of 68.1 kg leaps from a stationary hot air balloon. Derive a suitable mathematical model to compute velocity for the first 12 s of free fall. Also determine the terminal velocity that will be attained for an infinitely long cord. Use a drag coefficient of 0.25 kg/m.

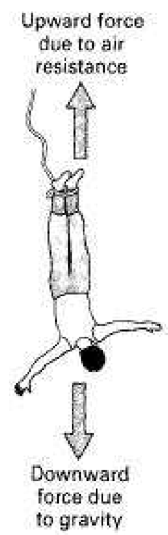

Figure 1: Forces acting on a free-falling bungee jumper.

2 Mathematical Modeling

The Newton's second law of motion forms the basis of mathematical modeling for this problem (**?**). Its mathematical expression is normally written as

$$F = ma \tag{1}$$

where F is the net force acting on the body (N or kg m/s^2), m is the mass of the object (kg) and a is its acceleration (m/s^2).

The second law can be recast in the format of

$$a = \frac{F}{m} \tag{2}$$

where a is the dependent variable reflecting the system's behaviour, F is the forcing function, and m is a parameter. Note that for this problem there is no independent variable because we are not yet predicting how acceleration varies in time or space.

The Newton's second law of motion can also be used to determine the terminal velocity of a free-falling body near the earth's surface. In our problem the falling body is the bungee jumper, Figure 1.

A mathematical model can be derived by expressing the acceleration as the time rate of change of the velocity (dv/dt), and substituting it into Eq. (2) to yield

$$\frac{dv}{dt} = \frac{F}{m} \tag{3}$$

where v (m/s) is velocity. Thus, the rate of change of the velocity is equal to the net force acting on the body normalized to its mass. If the net force is positive, the object will accelerate. If it is negative, the object will decelerate. If the net force is zero, the object's velocity will remain the same at a constant level.

We can now express the net force in terms of measurable variables and parameters. For a body falling within the vacinity of the earth, the net force is composed of two opposing forces: the downward pull of gravity F_D and the upward force of air resistance F_U, see Figure 1:

$$F = F_D + F_U \tag{4}$$

If force in the downward direction is assigned positive sign, the second law can be used to formulate the force due to gravity as

$$F_D = mg \tag{5}$$

where g is the acceleration due to gravity (9.81 m/s^2).
Air resistance is formulated from the science of fluid mechanics which suggests that a good approximation would be to assume that it is proportional to the square of the velocity,

$$F_U = -c_d v^2 \tag{6}$$

where c_d is the *drag coefficient* (kg/m). Thus, the greater the fall velocity, the greater the upward for due to air resistance. The parameter c_d account for properties of the falling object, such as shape or surface roughness, that affect air resistance. For this case, c_d might be a function of the type of clothing or the orientation used by the jumper during free fall.
The net force is the difference between the downward and upward force. Therefore, Eqs. (3) through (6) can be combined to yield

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2 \tag{7}$$

Eq. (7) is the mathematical model for this problem which relates the acceleration of a falling object to the forces acting on it. It is a *differential equation* because it is written in terms of the differential rate of change (dv/dt) of the variable that we are interested in predicting.

3 Solution to the Problem

3.1 Analytical Solution

The exact solution of Eq. (7) for the velocity of the jumper cannot be obtained using simple algebraic manipulation as in the solutiono of Eq. (2). If the jumper is initially at rest ($v = 0$ at $t = 0$), calculus can be used to solve Eq. (7) for

$$v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}} t\right) \tag{8}$$

where tanh is the hyperbolic tangent that can be either computed directly or via the more elementary exponential function as in

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{9}$$

Analytical solution to the problem may be obtained by inserting parameters into Eq. (8) to yield

$$v(t) = \sqrt{\frac{9.81 \times 68.1}{0.25}} \tanh\left(\sqrt{\frac{9.81 \times 0.25}{68.1}} t\right) = 51.6928 \tanh(0.18977t)$$

which can be used to compute the results in the table below:

Time t (s)	Velocity v (m/s)
0	0
2	18.7292
4	33.1118
6	42.0762
8	46.9575
10	49.4214
12	50.6175
∞	51.6938

A plot of velocity against time from this table is shown in Figure 2.

3.2 Numerical Solution

Numerical methods are those in which the mathematical problem is reformulated so it can be solved by arithmetic operations. The time rate of change of velocity for the mathematical model of this problem as represented by Eq. (7) can be approximated by [see Figure 1]:

$$\frac{dv}{dt} = \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} \tag{10}$$

where Δv and Δt are differences in velocity and time computed over finite intervals, $v(t_i)$ is velocity at an initial time t_i, and $v(t_{i+1})$ is velocity at some later time t_{i+1}. Note that $dv/dt \approx \Delta v/\Delta t$ is approximate because Δt is finite.
Remember from calculus,

$$\frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

Eq. (10) represents the reverse process and it can be substituted into Eq. (7) to give

$$\frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} = g - \frac{c_d}{m} v(t_i)^2$$

This equation can be rerranged to yield

$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c_d}{m} v(t_i)^2 \right] (t_{i+1} - t_i) \tag{11}$$

The term in brackets of the above equation is the RHS of the differential equation itself Eq. (3). That is, it provides a means to compute the rate of change or slope of v. Thus, the equation can be rewritten as

$$v_{i+1} = v_i + \frac{dv_i}{dt} \Delta t \tag{12}$$

where the nomenclature v_i designates velocity at time t_i and $\Delta t = t_{i+1} - t_i$. Eq. (12) can be solved using the Euler's method.

3.2.1 Algorithm

An appropriate algorithm may look like this:

1. Start
2. Assign values to known parameters: g, m, c_d
3. Compute value of velocity v for $t = 12$ s and display result
4. Set a period of time t over which the velocity is computed: from 0 to 30 s in steps of 2 s.
5. Compute value of velocity v for each value of t using Eq. (8)
6. Stop

You might want to refer **?**, Chapter 1 for a good guide on algorithm design.

3.2.2 Program Code

The Matlab script (?) to compute the velocity of the free-falling bungee jumper as given below:

```
% bungee.m
% Compute the free-fall velocity of an object
%
% Input:
%  g = acceleration due to gravity
%  m = mass
%  cd = drag coefficient
%
g = 9.81;
m = 68.1;
cd = 0.25;

% Compute velocity after 12 s
disp(' ')
fprintf('Velocity after 12 s is %8.4f m/s\n', ...
    sqrt(g*m/cd) * tanh(sqrt(g*cd/m)*12))

% Plot velocity againts time
t = [0:1:30]';
v = sqrt(g*m/cd) * tanh(sqrt(g*cd/m)*t);
plot(t,v,t,v,'o')
title('Plot of v versus t');
xlabel('Values of t')
ylabel('Values of v')
grid
```

was saved into the file `bungee.m` and run at the Matlab prompt thus:

```
>> bungee
```

The result was displayed as

```
Velocity after 12 s is  50.6175 m/s
```

together with the plot of velocity versus time. This plot of the results computed over a period of 30 s is shown in Figure 2.

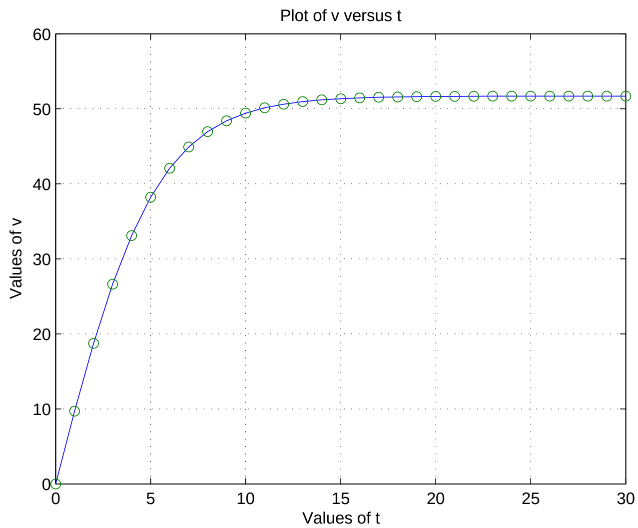

Figure 2: Plot of velocity versus time for free-falling body.

4 Conclusion

According to the model, Eq. (7), the jumper accelerates rapidly, Figure 2. A velocity of 50.6175 m/s is attained after 12 s. Note also that after a sufficiently long time, a constant velocity, called the *terminal velocity*, of 51.6983 m/s is reached. This velocity is constant because, eventually, the force of gravity will be in balance with the air resistance. Thus, the net force is zero and acceleration has ceased.

A Program Source Codes on CD

Attached herewith is a compact disc containing the Matlab source codes of the programs developed for this projects:

- `bungee.m`