Chapter 2

Heat Conduction Equation

Conduction Heat Transfer

- **Conduction refers to the transport of energy in a medium (solid, liquid or gas) due to a temperature gradient.**
- **The physical mechanism is random atomic or molecular activity**
- **Governed by Fourier's law**

- **In this chapter we will learn**
  - **The definition of important transport properties and what governs thermal conductivity in solids, liquids and gases**
  - **The general formulation of Fourier's law, applicable to any geometry and multiple dimensions**
  - **How to obtain temperature distributions by using the heat diffusion equation.**
  - **How to apply boundary and initial conditions**

Thermal Properties of Matter

- **Recall from Chapter 1, equation for heat conduction:**

$$q_x = k\frac{T_1 - T_2}{L} = k\frac{\Delta T}{L}$$

- **The proportionality constant is a transport property, known as thermal conductivity k (units W/m.K)**
- **Usually assumed to be isotropic (independent of the direction of transfer): $k_x = k_y = k_z = k$**

Is thermal conductivity different between gases, liquids and solids?

Thermal Conductivity (k) provides an indication of the rate at which energy is transferred by the diffusion process

Thermal Conductivity: Solids

- **Solid (metals) comprised of free electrons and atoms bound in lattice**
- **Thermal energy transported through**
  - **Migration of free electrons, k_e**
  - **Lattice vibrational waves, k_l**

$$k = k_e + k_l \qquad \text{where} \qquad k_e \approx \frac{1}{\text{electrical resistivity}, (\rho_e)}$$

? ***What is the relative magnitude in pure metals, alloys and non-metallic solids?***

Thermal Conductivity: Fluids

- **Intermolecular spacing is much larger**
- **Molecular motion is random**
- **Thermal energy transport less effective than in solids; thermal conductivity is lower**

➢ **Kinetic theory of gases:**

$$k \propto n\bar{c}\lambda$$

where n the number of particles per unit volume, $\bar{c}$ the mean molecular speed and l the mean free path (average distance travelled before a collision)

? ***What are the effects of temperature, molecular weight and pressure?***

Thermal Conductivity: Fluids

- **Physical mechanisms controlling thermal conductivity not well understood in the liquid state**
- **Generally k decreases with increasing temperature (exceptions glycerine and water)**
- **k decreases with increasing molecular weight.**
- **Values tabulated as function of temperature.**

Thermal Conductivity: Insulators

? *How can we design a solid material with low thermal conductivity?*

- **Can disperse solid material throughout an air space – fiber, powder and flake type insulations**
- **Cellular insulation – Foamed systems**

- **Several modes of heat transfer involved (conduction, convection, radiation)**
- **Effective thermal conductivity: depends on the thermal conductivity and radiative properties of solid material, volumetric fraction of the air space, structure/morphology (open vs. closed pores, pore volume, pore size etc.) Bulk density (solid mass/total volume) depends strongly on the manner in which the solid material is interconnected.**

In general, high **k** values are good conductors while low **k** values are better insulators.

k – <u>high</u> for metals
k – <u>low</u> of plastics, glass, wood

Table – compiled from Appendix A

Material	**Temp (K)**	**k (W/m-K)**	**Material**	**Temp (K)**	**k (W/m-K)**
Brick	300	0.72	Air	300	0.026
Cork	300	0.039	Copper	300	401
Glass	300	1.4	Aluminum	300	237

More accurate relationship is:

$$K = K_o(1 + aT)$$

where '$\mathbf{K_o}$' and '**a**' are constants. 'a' can be negative or positive depending on the material.

Thermal Diffusivity

Thermophysical properties of matter:

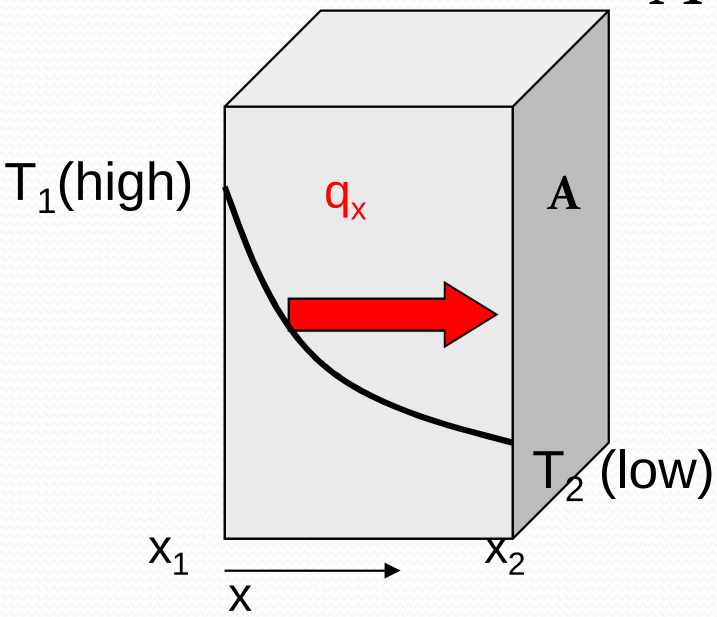
- **Transport properties: k (thermal conductivity/heat transfer), ν (kinematic viscosity/momentum transfer), D (diffusion coefficient/mass transfer)**
- **Thermodynamic properties, relating to equilibrium state of a system, such as density,ρ and specific heat c_p.**
  - **the volumetric heat capacity ρc_p (J/m³.K) measures the ability of a material to store thermal energy.**
- ***<u>Thermal diffusivity</u>* <u>α</u> is the ratio of the thermal conductivity to the heat capacity:**

$$\alpha = \frac{k}{\rho c_p}$$

The Conduction Rate Equation

Recall from Chapter 1:

- Heat rate in the x-direction $$Q_x = -kA\frac{dT}{dx}$$
- Heat flux in the x-direction $$q_x = \frac{q}{A} = -k\frac{dT}{dx}$$



We assumed that T varies only in the x-direction, T=T(x)

Direction of heat flux is normal to cross sectional area A, where A is isothermal surface (plane normal to x-direction)

The Conduction Rate Equation

In reality we must account for heat transfer in three dimensions

- Temperature is a scalar field T(x,y,z)
- Heat flux is a vector quantity. In Cartesian coordinates:

$$\underline{\mathbf{q}}_x = \underline{\mathbf{i}}q_x + \underline{\mathbf{j}}q_y + \underline{\mathbf{k}}q_z$$

for isotropic medium $$q_x = -k\frac{\partial T}{\partial x}, q_y = -k\frac{\partial T}{\partial y}, q_z = -k\frac{\partial T}{\partial z}$$

$$\therefore \underline{\mathbf{q}} = -k\left(\underline{\mathbf{i}}\frac{\partial T}{\partial x} + \underline{\mathbf{j}}\frac{\partial T}{\partial y} + \underline{\mathbf{k}}\frac{\partial T}{\partial z}\right) = k\underline{\nabla}T$$

Where three dimensional del operator in cartesian coordinates:

$$\underline{\nabla} = \underline{\mathbf{i}}\frac{\partial}{\partial x} + \underline{\mathbf{j}}\frac{\partial}{\partial y} + \underline{\mathbf{k}}\frac{\partial}{\partial z}$$

Summary: Fourier's Law

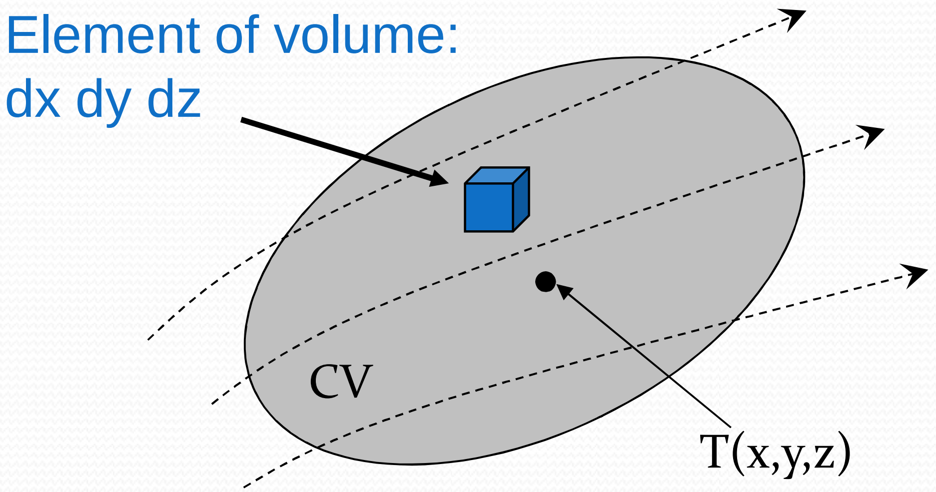
- It is phenomenological, ie. based on experimental evidence
- Is a vector expression indicating that the heat flux is normal to an isotherm, in the direction of decreasing temperature
- Applies to all states of matter
- Defines the thermal conductivity, ie.

$$k \equiv -\frac{q_x}{(\partial T / \partial x)}$$

The Heat Diffusion Equation

- Objective to determine the temperature field, ie. temperature distribution within the medium.
- Based on knowledge of temperature distribution we can compute the conduction heat flux.

➢ Reminder from fluid mechanics: Differential control volume.



We will apply the energy conservation equation to the differential control volume

reminder…

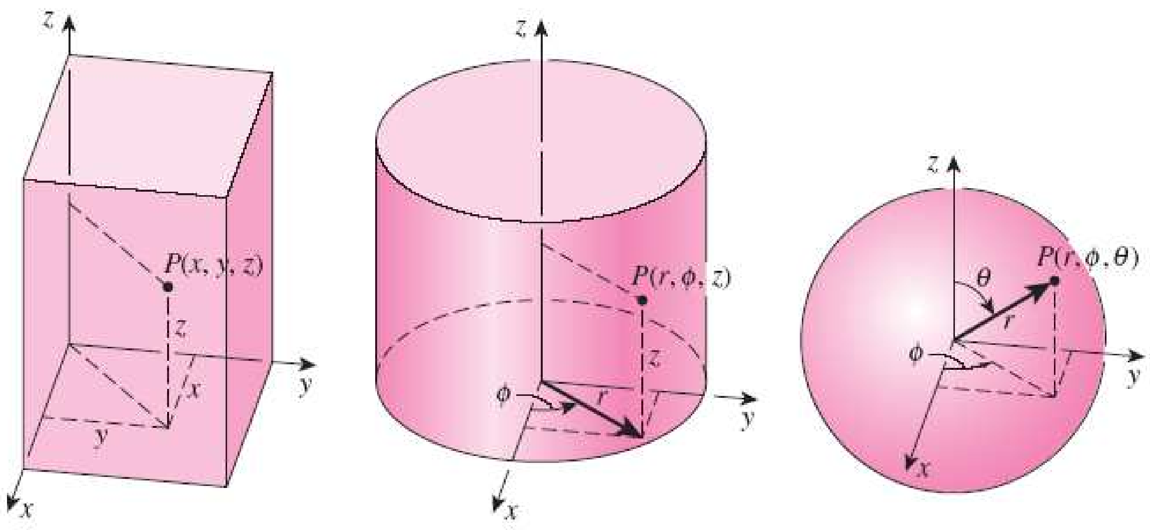
DO NOT confuse or interchange the term & units of :

Quantity	Meaning	Symbol	Units
Thermal Energy+	Energy associated with microscopic behavior of matter	U or u	J or J/kg
Temperature	A means of indirectly assessing the amount of thermal energy stored in matter	T	K or °C
Heat Transfer	Thermal energy transport due to temperature gradients		
Heat	Amount of thermal energy transferred over a time interval $\Delta t > 0$	Q	J
Heat Rate	Thermal energy transfer per unit time	$\dot{Q}$	J/s or W
Heat Flux	Thermal energy transfer per unit time and surface area	q''	W/m^2

+
$U \rightarrow$ Thermal energy of system
$u \rightarrow$ Thermal energy per unit mass of system

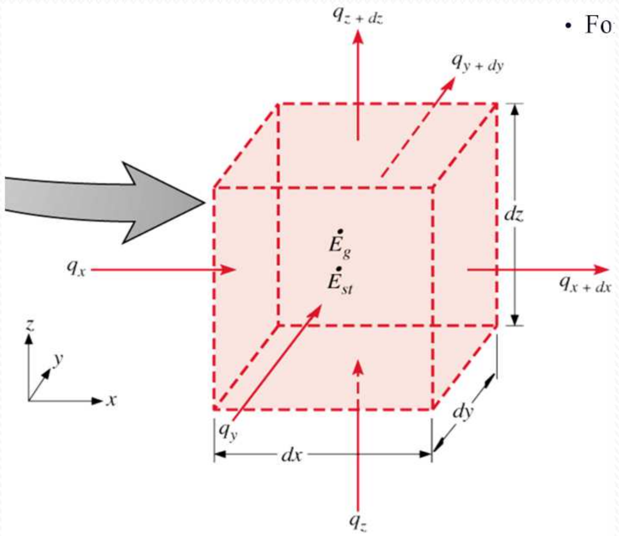
Chapter 2 : Introduction to Conduction

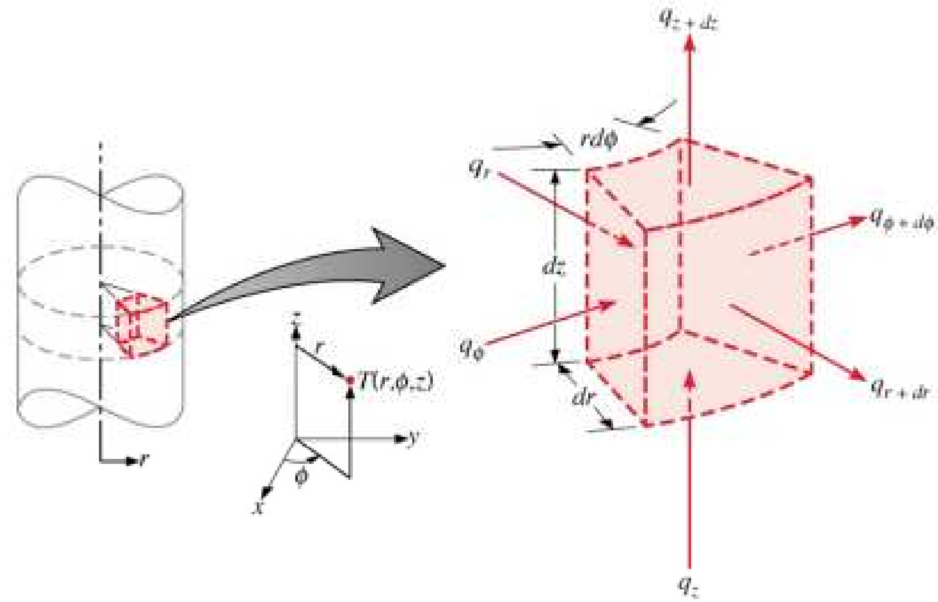
- The driving force for any form of heat transfer is the ***temperature difference***.
- The larger the temperature difference, the larger the rate of heat transfer.
- Three prime coordinate systems:
  - **Cartesian/rectangular $T(x, y, z)$**
  - **cylindrical $T(r, \phi, z)$**
  - **spherical $T(r, \phi, \theta)$.**

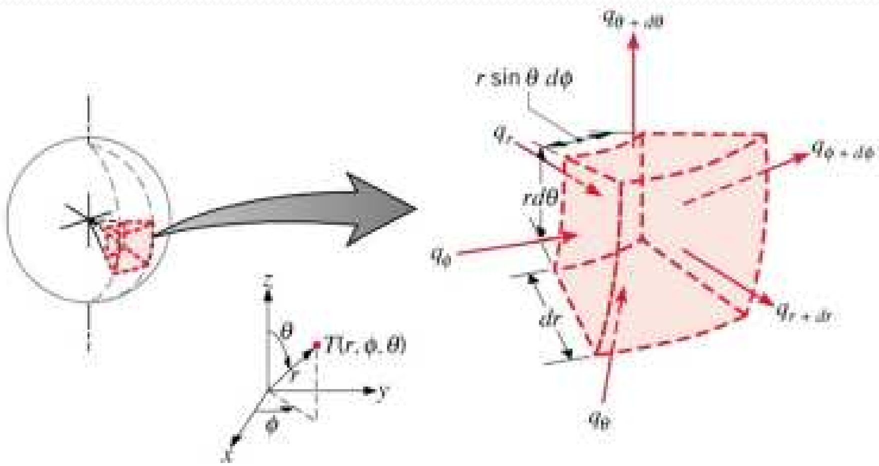


(a) Rectangular coordinates (b) Cylindrical coordinates (c) Spherical coordinates

Differential volumes







- Heat transfer problems are also classified as being:
  - ***one-dimensional***
  - ***two dimensional***
  - ***three-dimensional***
- In the most general case, heat transfer through a medium is **three-dimensional.** However, some problems can be classified as two- or one-dimensional depending on the relative magnitudes of heat transfer rates in different directions and the level of accuracy desired.
- **One-dimensional** if the temperature in the medium varies in **one direction only** and thus heat is transferred in one direction, and the variation of temperature and thus heat transfer in other directions are negligible or zero.
- **Two-dimensional** if the temperature in a medium, in some cases, varies mainly in **two primary directions**, and the variation of temperature in the third direction (and thus heat transfer in that direction) is negligible.

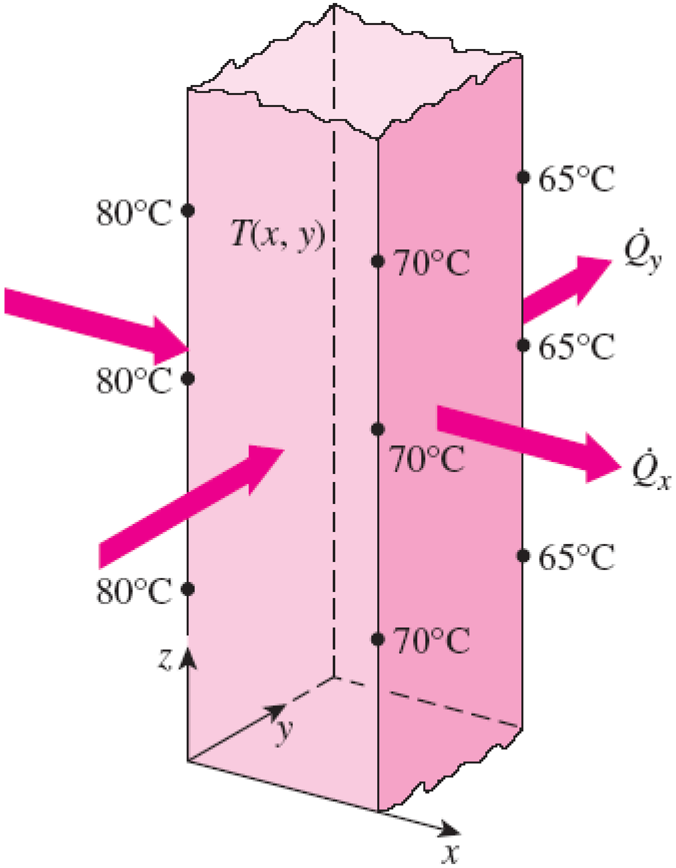


FIGURE 2–5

Two-dimensional heat transfer in a long rectangular bar.

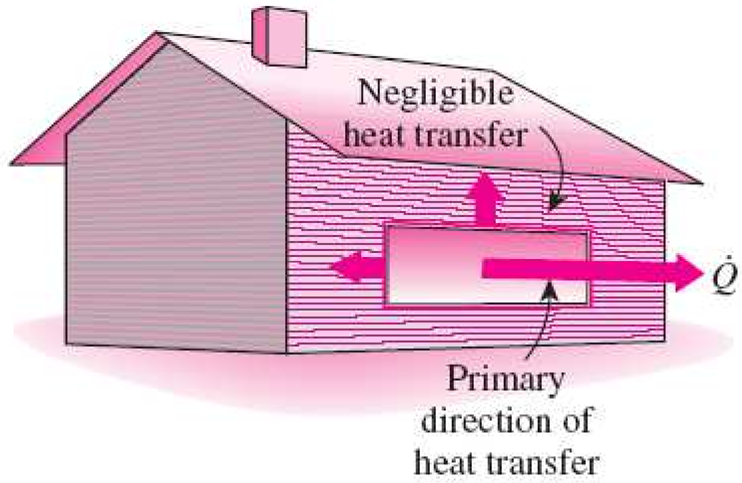


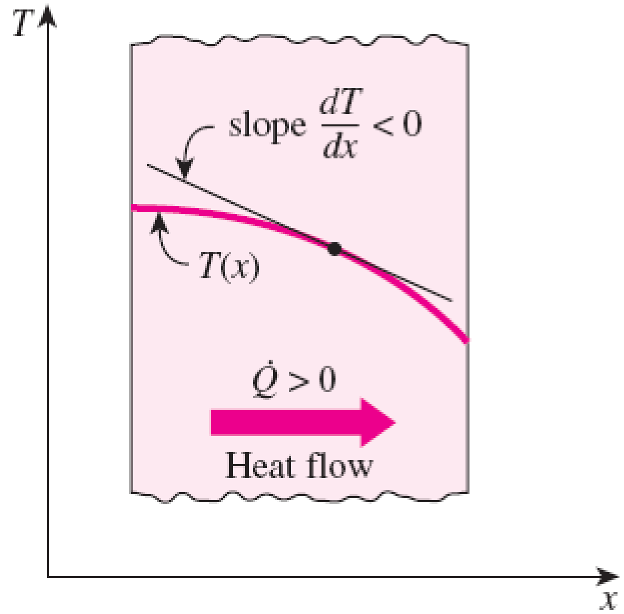
FIGURE 2–6

Heat transfer through the window of a house can be taken to be one-dimensional.

2.1 The conduction rate equation : Fourier's Law

- The rate of heat conduction through a medium in a specified direction (say, in the *x*-direction) is expressed by **Fourier's law of heat conduction** for one-dimensional heat conduction as:

$$\dot{Q}_{\text{cond}} = -kA\frac{dT}{dx} \qquad \text{(W)}$$



The temperature gradient dT/dx is simply the slope of the temperature curve on a T-x diagram.

Heat is conducted in the direction of decreasing temperature, and thus the temperature gradient is negative when heat is conducted in the positive *x*-direction.

2.1 The conduction rate equation : Fourier's Law

*for constant value of *k*

$$q'' \propto \left(-\frac{dT}{dx}\right)$$

- A rate equation that allows determination of the conduction heat flux from knowledge of the temperature distribution in a medium
- Its most general (vector) form for multidimensional conduction is

$$\vec{q}'' = -k\vec{\nabla} T \qquad \longrightarrow \qquad (2.3)$$

where, T(x, y, z) is the scalar temperature and ∇ is the 3-D del operator

Implications:

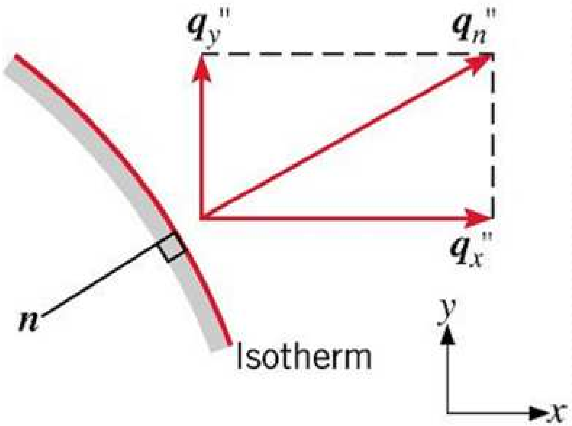
-Heat transfer is in the direction of decreasing temperature (basis for minus sign)

- Fourier's Law serves to define the thermal conductivity of the medi $\left(k \equiv -\vec{q}'' / \vec{\nabla} T\right)$

- Heat flux vector may be resolved into orthogonal components

- Cartesian Coordinates: $T(x,y,z)$

$$q_n'' = -k\vec{\nabla}T = -k\underbrace{\frac{\partial T}{\partial x}\vec{i}}_{q_x''} - k\underbrace{\frac{\partial T}{\partial y}\vec{j}}_{q_y''} - k\underbrace{\frac{\partial T}{\partial z}\vec{k}}_{q_z''}$$

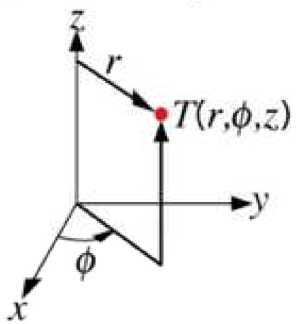


where: $T(x,y,z)$ is the scalar temperature

- Direction of heat transfer is <u>perpendicular</u> to lines of constant temperature (isotherms)

- The equation also can be expressed in other coordinates system

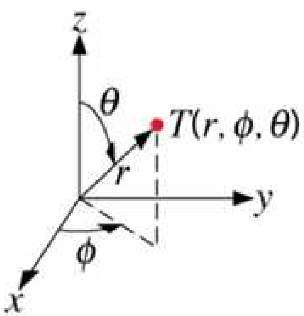
(1)



- Cylindrical Coordinates: $T(r,\phi,z)$

$$\vec{q''} = \underbrace{-k\frac{\partial T}{\partial r}\vec{i}}_{q''_r} \underbrace{-k\frac{\partial T}{r\partial \phi}\vec{j}}_{q''_\phi} \underbrace{-k\frac{\partial T}{\partial z}\vec{k}}_{q''_z}$$

(2)



- Spherical Coordinates: $T(r,\phi,\theta)$

$$\vec{q''} = \underbrace{-k\frac{\partial T}{\partial r}\vec{i}}_{q''_r} \underbrace{-k\frac{\partial T}{r\partial \theta}\vec{j}}_{q''_\theta} \underbrace{-k\frac{\partial T}{r\sin\theta\,\partial \phi}\vec{k}}_{q''_\phi}$$

- In angular coordinates (ϕ or ϕ, θ), the temperature gradient is still based on temperature change over a length scale and hence has units of K/m and not K/deg .

- For example, the heat rate for one dimensional, radial conduction in a cylinder or sphere is:

– Cylinder

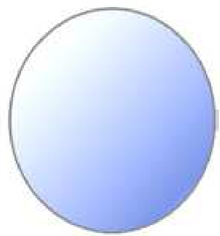
$$q_r = A_r q_r'' = 2\pi r L q_r''$$

or,

$$q_r' = A_r' q_r'' = 2\pi r q_r''$$

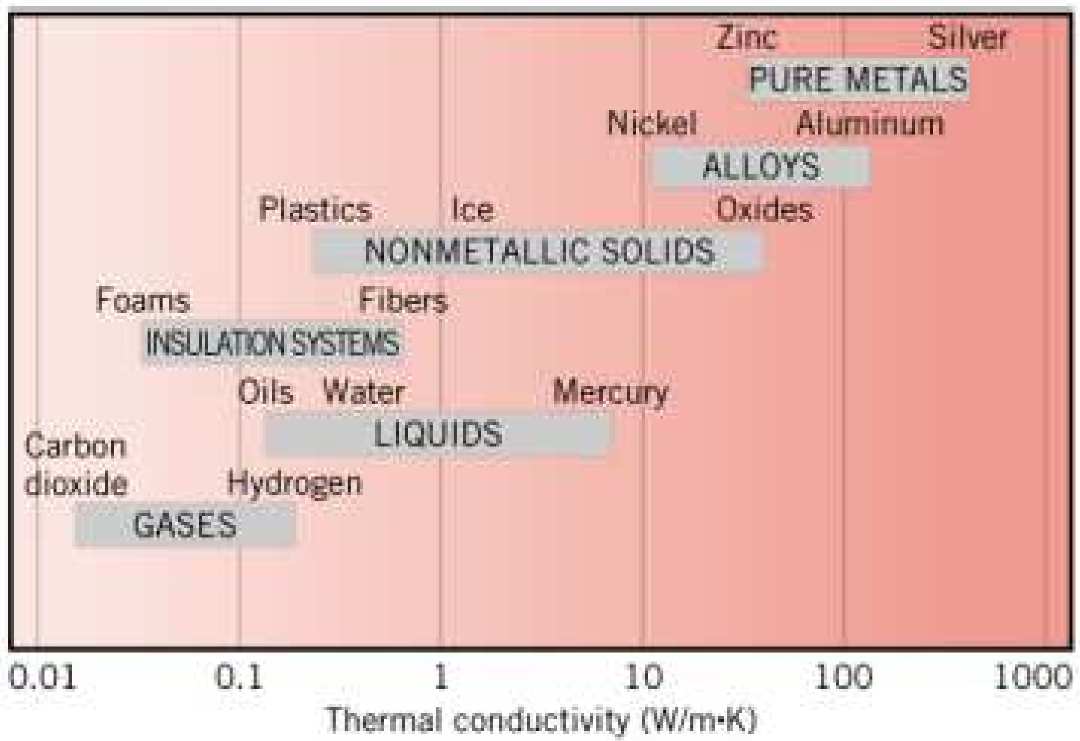
– Sphere

$$q_r = A_r q_r'' = 4\pi r^2 q_r''$$



2.2 Thermal properties

- **Thermal conductivity** – a measure of a material's ability to transfer heat by conduction



2.2 Thermal properties

- **Thermal diffusivity** – a measure of a material's ability to respond to changes in its thermal environment

- Physically it represents how fast heat diffuses through a material

$$\alpha = \frac{\text{Heat conduction}}{\text{Heat storage}} = \frac{k}{\rho c_p} \qquad (\mathrm{m^2/s})$$

- Further information – refer to property tables :

Solids: Tables A.1 – A.3
Gases: Table A.4
Liquids: Tables A.5 – A.7

The thermal diffusivities of some materials at room temperature

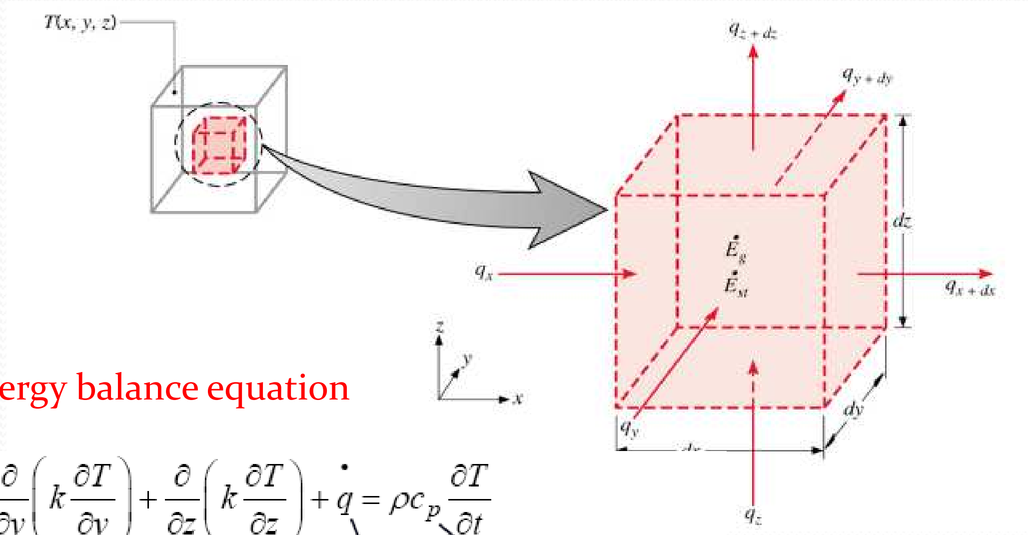
Material	α, m²/s*
Silver	149×10^{-6}
Gold	127×10^{-6}
Copper	113×10^{-6}
Aluminum	97.5×10^{-6}
Iron	22.8×10^{-6}
Mercury (l)	4.7×10^{-6}
Marble	1.2×10^{-6}
Ice	1.2×10^{-6}
Concrete	0.75×10^{-6}
Brick	0.52×10^{-6}
Heavy soil (dry)	0.52×10^{-6}
Glass	0.34×10^{-6}
Glass wool	0.23×10^{-6}
Water (l)	0.14×10^{-6}
Beef	0.14×10^{-6}
Wood (oak)	0.13×10^{-6}

2.3 The heat equation

- A differential equation whose solution provides the temperature distribution in a stationary medium.

- Based on applying conservation energy to a differential control volume through which energy transfer is exclusively by conduction.

- For example, the heat equation for Cartesian coordinates is



- Using energy balance equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

Net transfer of thermal energy into the control volume (inflow-outflow) | Thermal energy generation | Change in thermal energy storage

Energy Balance

$$\underbrace{Q_{in} - Q_{out}}_{\text{Net heat transfer}} + \underbrace{E_{gen}}_{\text{Heat generation}} = \underbrace{\Delta E_{thermal,\ system}}_{\text{Change in thermal energy of the system}} \qquad \text{(J)}$$

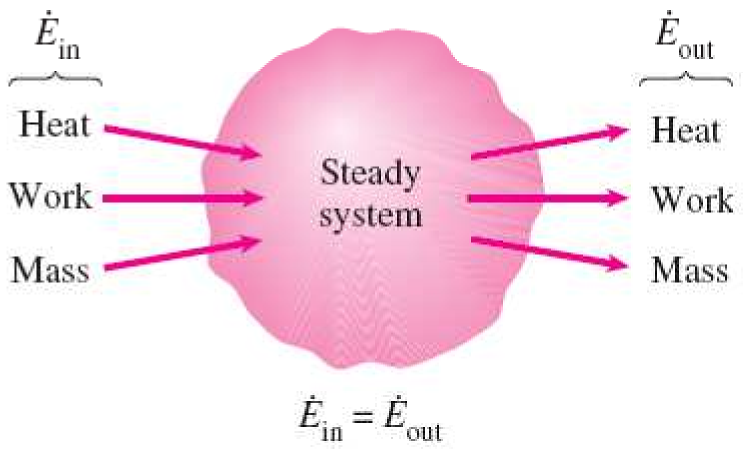


FIGURE 1–15

In steady operation, the rate of energy transfer to a system is equal to the rate of energy transfer from the system.

In heat transfer problems, it is convenient to write a heat balance and to treat the conversion of nuclear, chemical, mechanical, and electrical energies into thermal energy as *heat generation*.

Surface Energy Balance

A surface contains no volume or mass, and thus no energy. Therefore, a surface can be viewed as a fictitious system whose energy content remains constant during a process.

Surface energy balance: $\dot{E}_{in} = \dot{E}_{out}$

This relation is valid for both steady and transient conditions, and the surface energy balance does not involve heat generation since a surface does not have a volume.

$$\dot{Q}_1 = \dot{Q}_2 + \dot{Q}_3$$

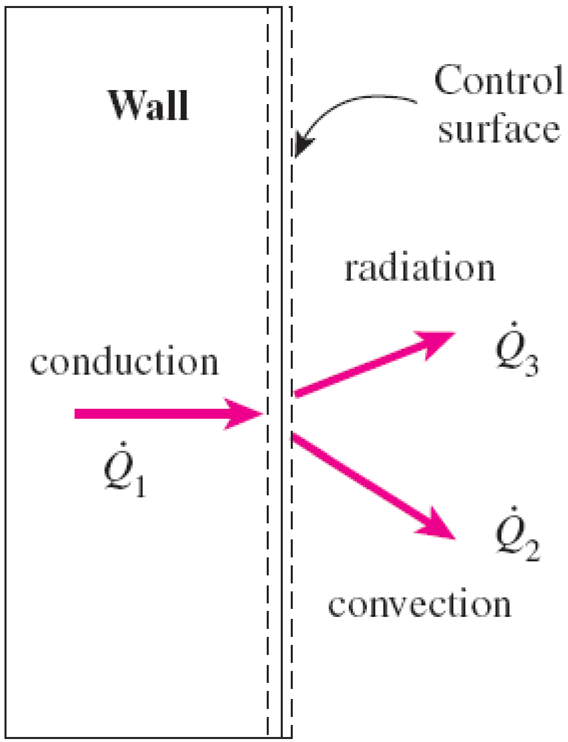


FIGURE 1–19
Energy interactions at the outer wall surface of a house.

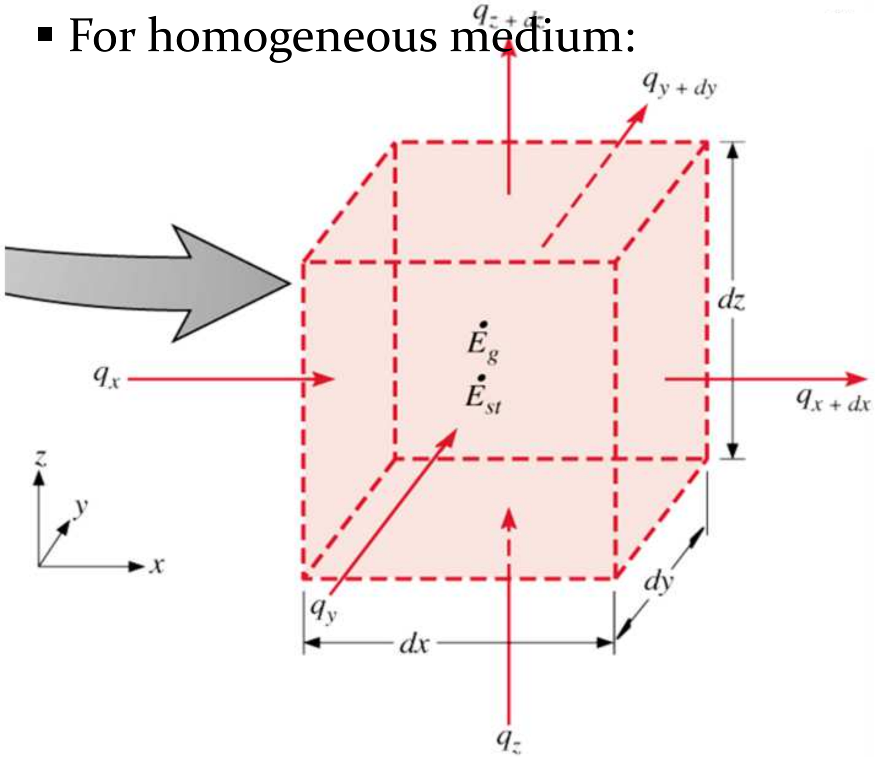
When the directions of interactions are not known, all energy interactions can be assumed to be towards the surface, and the surface energy balance can be expressed as $\Sigma \dot{E}_{in} = 0$. Note that the interactions in opposite direction will end up having negative values, and balance this equation.

The heat equation (Cartesian Coordinates)

- Applying **conservation of energy** to a infinitely small differential control volume at an instant in time through which energy transfer is by conduction only

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

- For homogeneous medium:



$$q_x = -k(dy \cdot dz)\frac{\partial T}{\partial x}$$

$$q_y = -k(dx \cdot dz)\frac{\partial T}{\partial y}$$

$$q_z = -k(dx \cdot dy)\frac{\partial T}{\partial z}$$

q_{in}

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} \cdot dx + \frac{1}{2}\frac{\partial^2 q_x}{\partial x^2} \cdot \partial x^2 + \ldots$$

Neglecting high order terms

$$q_{x+dx} \simeq q_x + \frac{\partial q_x}{\partial x} \cdot dx$$

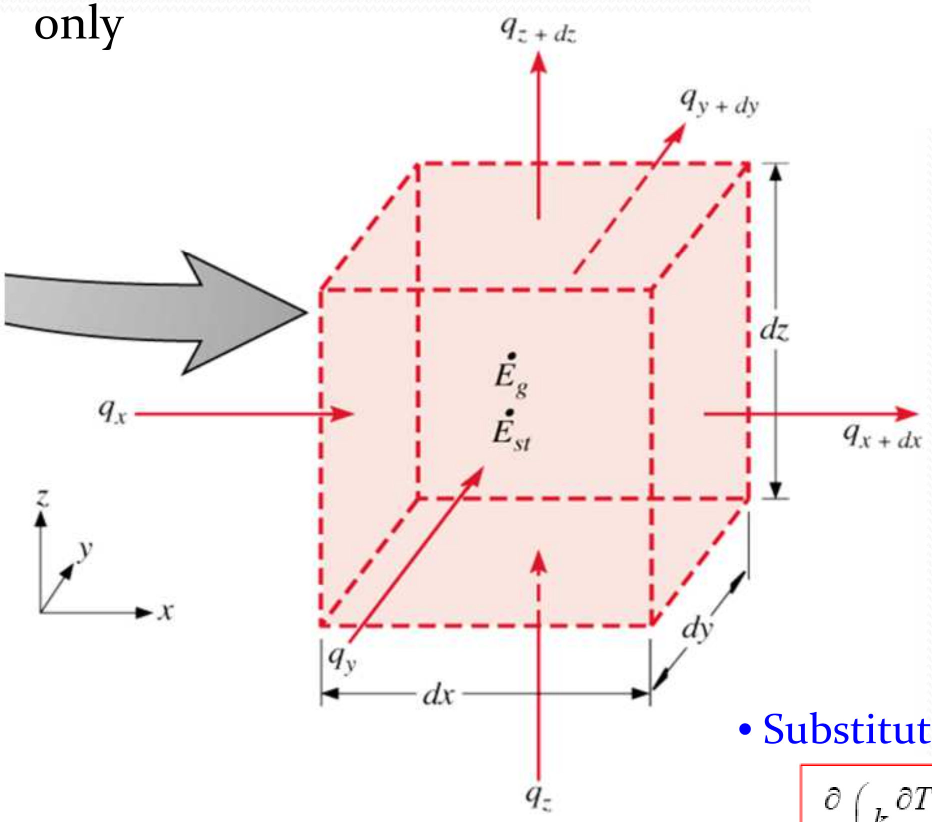
$$q_y \simeq q_y + \frac{\partial q_y}{\partial y} \cdot dy$$

$$q_z'' \simeq q_z + \frac{\partial q_z}{\partial z} \cdot dz$$

q_{out}

The heat equation (Cartesian Coordinates)

- Applying conservation of energy to a infinitely small differential control volume at an instant in time through which energy transfer is by conduction only



- The energy source term

$$\dot{E}_g = \dot{q}\cdot(dx\cdot dy\cdot dz)$$

- The energy storage term

$$\dot{E}_{st} = \rho\cdot C_p\cdot\frac{\partial T}{\partial t}(dx\cdot dy\cdot dz)$$

if $+ve$ then it is a source term

if $-ve$ then it is a sink term

- An energy balance gives

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

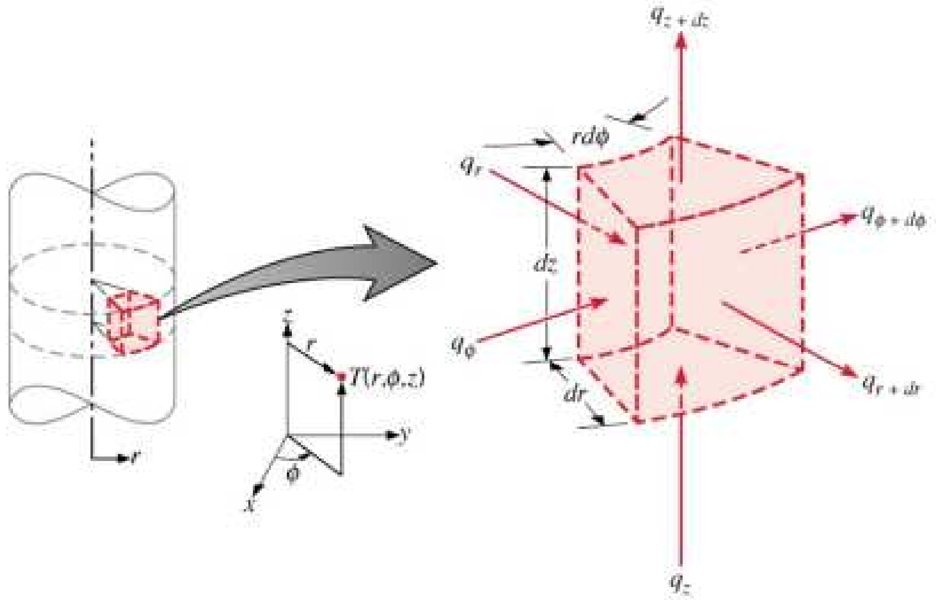
$$q_{in} - q_{out} + \dot{q}_g\cdot(dx\cdot dy\cdot dz) = \dot{E}_{st}$$

- Substituting gives

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \quad (2.17)$$

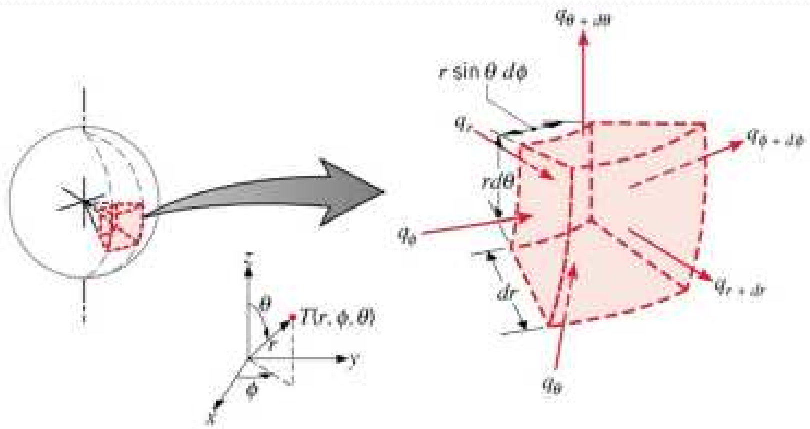
-which is the general form of the Heat diffusion equation in Cartesian Coordinates

- For cylindrical coordinates



$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \quad \rightarrow (2.24)$$

- For spherical coordinates



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

→ (2.27)

Boundary and Initial Conditions

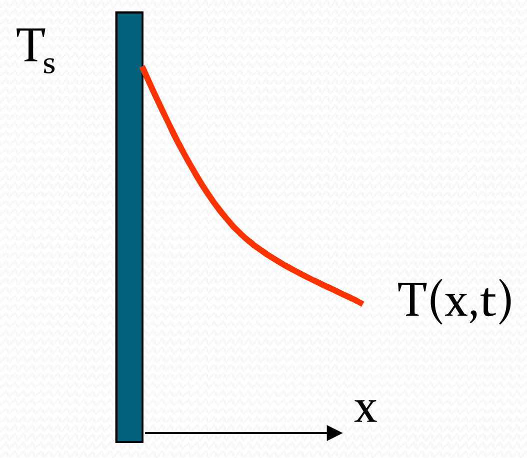
- Heat equation is a differential equation:
  - Second order in spatial coordinates: Need 2 boundary conditions
  - First order in time: Need 1 initial condition

Boundary Conditions

1) FIRST KIND (DIRICHLET CONDITION):

Prescribed temperature

Example: a surface is in contact with a melting solid or a boiling liquid

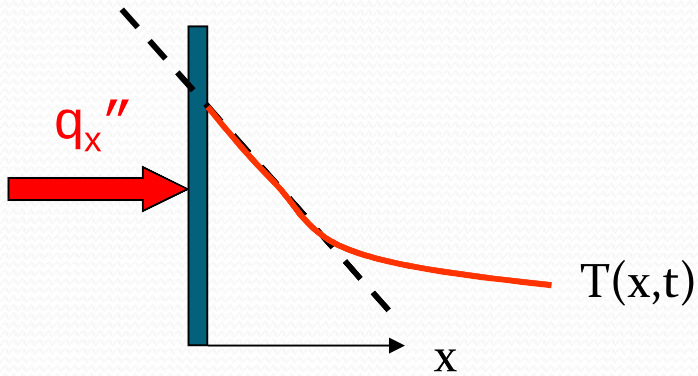


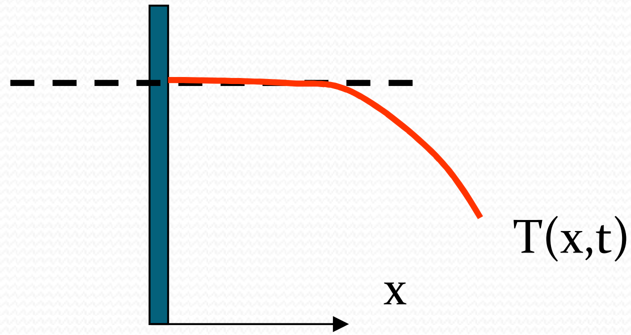
Boundary and Initial Conditions

2) SECOND KIND (NEUMANN CONDITION):

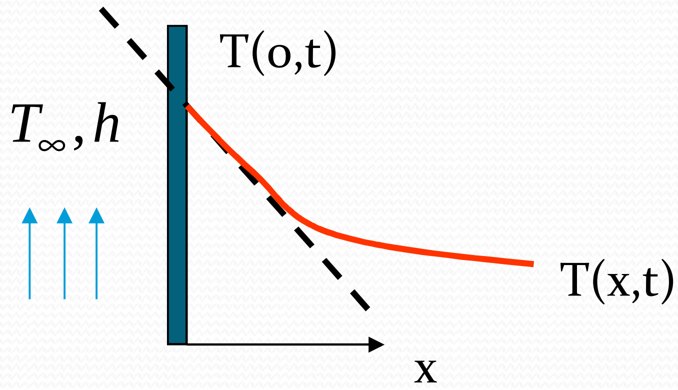
Constant heat flux at the surface

Example: What happens when an electric heater is attached to a surface? What if the surface is perfectly insulated?

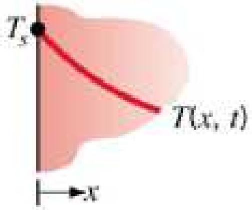




3) THIRD KIND (MIXED BOUNDARY CONDITION) :
When *convective* heat transfer occurs at the surface

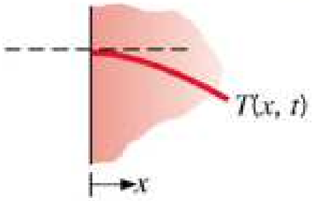


Constant Surface Temperature:



$$T(0,t) = T_s$$

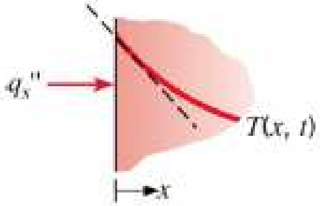
Insulated Surface



$$\frac{\partial T}{\partial x}\Big|_{x=0} = 0$$

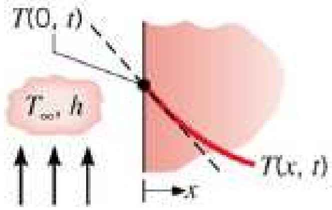
Constant Heat Flux:

Applied Flux



$$-k\frac{\partial T}{\partial x}\Big|_{x=0} = q_s''$$

Convection



$$-k\frac{\partial T}{\partial x}\Big|_{x=0} = h\left[T_\infty - T(0,t)\right]$$

Example: (Prob 2.23)

The steady-state temperature distribution in a one-dimensional wall of thermal conductivity 50 W/mK and thickness 50 mm is observed to be T(°C) = $a + bx^2$, where a = 200°C , b = -2000°C/m^2, and x is in meters.

i) What is the generation rate, $\dot{q}$ in the wall ?

ii) Calculate the heat fluxes at the two wall faces. In what manner are these heat fluxes related to the heat generation rate ?

Example: (Prob 2.24)

The temperature distribution across a wall 0.3 m thick at a certain instant of time is $T(x) = a + bx + cx^2$, where T is in degree Celcius and x is in meters, a = 200°C, b = -200°C/m and c = 30°C/m². The wall has a thermal conductivity of 1 W/mK.

i) On a unit surface area basis, determine the rate of heat transfer (heat flux) into and out of the wall and the rate of change of energy stored by the wall.

ii) If the cold surface is exposed to a fluid at 100°C, what is the convection coefficient

Example: (Prob 2.26)

One dimensional, steady state conduction with uniform internal energy generation occurs in a plane wall with a thickness of 50 mm and a constant thermal conductivity of 5 W/mK. For these conditions, the temperature distributions has the form , $T(x) = a + bx + cx^2$. The surface at x=0 has a temperature of 120°C and experiences convection with a fluid for which T_∞=20°C and h=500W/m²K. The surface at x=L is well insulated.

i) Applying an overall energy balance to the wall, calculate the internal energy generation rate.

ii) Determine the coefficients a, b and c by applying the boundary conditions to the prescribed temperature distribution.

Example: (Prob 2.40)

Two-dimensional steady state conduction occurs in a hollow cylindrical solid of thermal conductivity, k = 16 W/mK, outer radius, r_o=1m, and overall length, $2z_o$=5m, where the origin of the coordinate system is located at the midpoint of the centerline. The inner surface of the cylinder is insulated, and the temperature distribution within the cylinder has the form $T(r,z) = a + br^2 + c\ln r + dz^2$ where a=20°C, b=150°C/m², c=-12°C, d=-300°C/m² and r and z are in meters.

a) Determine the inner radius, r_i of the cylinder

b) Obtain an expression for the volumetric rate of heat generation

c) Determine the axial distribution of the heat flux at the outer surface , $q''_r(r_o,z)$. What is the heat rate at the outer surface? Is it into or out of the cylinder?

d) Determine the radial distribution of the heat flux at the both end faces of the cylinder, $q''_z(r,+z_o)$ and $q''_z(r,-z_o)$. What are the corresponding heat rates ? Are they into or out of the cylinder ?

e) Verify that your results are consistent with an overall energy balance on the cylinder